

Relation between two approaches of the colour hydrodynamics

Abhishek Basak and Jitesh R. Bhatt*

Theoretical Physics Division, Physical Research Laboratory, Ahmedabad-380009, India

Predhiman K. Kaw

Institute for Plasma Research, Bhat, Gandhinagar, India

It is argued that the short time scale phenomena can be studied within the framework of hydrodynamics in the quark-gluon plasma. There are two different versions of the hydrodynamic-like equations in the literature. In this work we discuss the possible relationship between these versions. In particular we show that if the colour charges associated with the velocity and density matrices in the matrix version of hydrodynamics are same then both the versions of the hydrodynamics become identical.

Due to its simplicity over the other descriptions of the electromagnetic plasma, the hydrodynamic approach is extremely useful in describing many bulk and collective properties of the plasma. In fact the equations of hydrodynamics have been successfully employed to study variety of non-linear and non-equilibrium phenomena in electromagnetic plasma [1]. However, in the literature of quark-gluon plasma (QGP), the application of hydrodynamics in studying the collective and bulk-properties of the plasma have not been fully exploited [2]. The application of hydrodynamics is generally restricted to investigate the evolution of thermalized QGP only. It should be noted here that there are certain issues related with the application of the hydrodynamics to the thermalized QGP still remains [3] like inclusion of viscous effects, formation time etc. On the other hand the transport theory is employed to investigate the properties of non-equilibrium QGP [4]. It must be emphasized that in the past, inspired by the success of hydrodynamics for electromagnetic plasmas, the non-equilibrated QGP related issues like two-stream instability and thermalization of the plasma were studied using the chromohydrodynamics(CHD) equations [5]. The linear perturbative analysis done in Ref.[5] was shown to give the results similar to that obtained from the kinetic theory [4]. Moreover, the hydrodynamics approach was also used to study various collective phenomena in QGP [6–13]. Usually in hydrodynamics one introduces a length L and a time τ which characterize distance and time over which the plasma quantities change significantly. For the validity of hydrodynamics length associated with the fluid element ΔV , should be satisfying $(\Delta V)^{1/3} \ll L$ and $(\Delta V)^{1/3} \gg \lambda$, where λ is the mean-free path. This allows the fluid elements to persists over a several mean-free paths or for a several collision frequencies. However in non-equilibrium plasmas for collective modes in collisionless limit another condition is more relevant: In general each fluid element will have random velocity $v_{thermal}$ and a flow velocity component U . If U is same for all

particles (if the fields acting on them are same) then the fluid element can persist if wave-vector k and frequency ω satisfy the condition $kv_{thermal} \ll \omega$. Thus in a few wave periods thermal effects do not spread the particles apart by as much as a wavelength and the fluid element essentially stays in tact, bound by collective fields. This is quite unlike a neutral gas where there is no long-range self-consistent field which can hold the fluid elements in cold-collision less limit and therefore in that case hydrodynamics might be meaningless (see for example Ref.[14]). Thus hydrodynamical equations has been applied to study various collective phenomena in both QED and quark-gluon-plasma [1, 2, 10]. However, hydrodynamics approach can be inadequate to describe certain non-equilibrium phenomena where the momentum/velocity description of the particle play an important role [1]. At least in the linear regime, for a quark-gluon plasma, it is found that the dispersion-relation for the collective modes calculated either using kinetic-theory or hydrodynamics remain same if the velocity dependent phenomena like the Landau damping are ignored.

At present there are two versions of CHD equations in the literature of quark-gluon plasma [2, 6]. Therefore it is important, in our opinion, to look for the possible relationship between these two versions of the hydrodynamics. This kind of study may be helpful in understanding the assumptions under which the given set of CHD equations is valid. This may in turn help in choosing an appropriate hydrodynamical equations for a given physical problem.

In what follows, we first briefly review the two versions of CHD equations given in Refs. [2, 6]: CHD equations were obtained from the QCD Lagrangian with some heuristic arguments in Ref.[6]. Later more systematic derivation using the kinetic-theory was given in Ref.[15]. However in Ref.[15] the application to the pre-equilibrium phenomena was not discussed. These equa-

tions are give below:

$$\partial_\mu N_\alpha^\mu = 0, \quad (1)$$

$$\partial_\mu T_\alpha^{\mu\nu} = -g j_{\alpha\mu}^b F_b^{\mu\nu}, \quad (2)$$

$$U_\alpha^\mu \partial_\mu I_{\alpha a} = -f_{abc} U_\alpha^\mu A_{\mu b} I_{\alpha c}, \quad (3)$$

where, the index α denotes a stream or a specie (quark, anti-quark, gluons etc), $N_\alpha^\mu = \tilde{n}_\alpha U_\alpha^\mu$ is postulated to be the flux of specie α in a way similar to [2]. The four hydrodynamical velocity U_α^μ is a colour singlet it can be written in terms of three velocities v_α as $\gamma_\alpha (1, v_\alpha)$ with $\gamma_\alpha = 1/\sqrt{1-v_\alpha^2}$. The quantity Q_α transforms covariantly under the gauge transformations and it can be regarded as a non-abelian "colour-charge" of the fluid element. It is equation (3) that makes the CHD equations of Ref.[6] completely different from the equations of electromagnetic plasmas. Finally the energy momentum tensor $T_\alpha^{\mu\nu}$ can be written as follows

$$T_\alpha^{\mu\nu} = (\tilde{\epsilon}_\alpha + \tilde{p}_\alpha) U_\alpha^\mu U_\alpha^\nu - \tilde{p}_\alpha g^{\mu\nu} \quad (4)$$

where $\tilde{\epsilon}_\alpha$ and \tilde{p}_α are energy density and pressure for the specie a . It must be noted that in this approach all the hydrodynamical variables namely $\tilde{n}_\alpha, U_\alpha^\mu, \tilde{\epsilon}_\alpha, \tilde{p}_\alpha$ are considered to be the colour scalars except I_α . Equations 3 need to be supplemented by the equation of state and Yang-Mills equations. The current density j^μ can be defined using the hydrodynamical variables as follows

$$j_\alpha^\mu = g \Sigma_\alpha I_\alpha n_\alpha U_\alpha^\mu. \quad (5)$$

The alternative formulation of the CHD equations [2] regard the hydrodynamical variables as $N \times N$ -matrices for $SU(N)$ -colour group. In this approach we have the following set of CHD equations:

$$D_\mu n_\alpha^\mu = 0, \quad (6)$$

$$D_\mu t_\alpha^{\mu\nu} = \frac{g}{2} \{F_\mu^\nu, n_\alpha^\mu\} \quad (7)$$

where, α labels the stream or species and the flux n_α^μ is defined as $n_\alpha^\mu = n_\alpha u_\alpha^\mu$. $D_\mu = \partial_\mu - ig[A_\mu, \dots]$ is the covariant derivative and $\{\dots, \dots\}$ denote the anti-commutator. The energy-momentum tensor $T_\alpha^{\mu\nu}$ for this version of CHD differs from the equation (4) and it is defined as

$$t_\alpha^{\mu\nu} = \frac{1}{2} (\epsilon_\alpha + p_\alpha) \{u_\alpha^\mu, u_\alpha^\nu\} - p_\alpha g^{\mu\nu}. \quad (8)$$

The current density in the fundamental representation defined as

$$j^\mu = -\frac{g}{2} \Sigma_\alpha \left(n_\alpha u_\alpha^\mu - \frac{1}{N} \text{tr} [n_\alpha u_\alpha^\mu] \right) \quad (9)$$

All the hydrodynamical variables namely $n_\alpha, u_\alpha^\mu, \epsilon$ and p , in this formulation, are $N \times N$ matrices for $SU(N)$ -group and transform covariantly under the gauge transformations. Derivation of equations (6-7) from the kinetic theory with colour covariant quark and gluon distribution functions was given in Ref.[2].

Let us first note that irrespective of the version of the CHD equations used, the covariant continuity equation is always satisfied i.e.

$$D_\mu j^\mu = 0. \quad (10)$$

For simplicity let us consider the case for $SU(2)$ group and assume the 'cold' plasma $p_\alpha = 0$ and $\epsilon = m_\alpha n_\alpha$ with m_α being the mass of the particle in specis α . In this case we can write equation (8) as

$$t_\alpha^{\mu\nu} = \frac{m_\alpha n_\alpha}{2} \{u_\alpha^\mu, u_\alpha^\nu\} \quad (11)$$

Next, consider the following decomposition of the quantities n_α and u_α^μ appearing in equations (8-9),

$$n_\alpha = (I_\alpha + I_{\alpha 0}) n_{\alpha s}, \quad (12)$$

$$u_\alpha^\mu = (Q_\alpha + Q_{\alpha 0}) u_{\alpha s}^\mu \quad (13)$$

where, we have introduced the scalar quantities $n_{\alpha s}$ and $u_{\alpha s}^\mu$ for the gauge invariant number density and velocities for a given specie or a stream respectively. Quantities $(I_\alpha + I_{\alpha 0})$ and $(Q_\alpha + Q_{\alpha 0})$ transforms gauge covariantly and they can be represented as $(I_{\alpha a} T^a + I_{\alpha 0} \mathbb{I})$ and $(Q_{\alpha a} T^a + Q_{\alpha 0} \mathbb{I})$ respectively with T^a being the group generators and \mathbb{I} is an identity. It ought to be noted here that for gluon sector we need the adjoint representation as the gluon distribution function in the underlying kinetic theory description represented by $(N^2 - 1) \times (N^2 - 1)$ matrix. For the present purpose we restrict ourselves for the fundamental representations necessary for describing the quark sector only. Generalization for the gluonic sector can be straightforward [2]. The total current density can be constructed as a sum of the current generated by each specie i.e. $j^\mu = \Sigma_\alpha j_\alpha^\mu$ where,

$$j_\alpha^\mu = -\frac{g}{2} \left(n_\alpha u_\alpha^\mu - \frac{1}{N} \text{tr} [n_\alpha u_\alpha^\mu] \right). \quad (14)$$

From the above definitions one can always write,

$$n_\alpha u_\alpha^\mu = \mathcal{R}_\alpha(IQ) n_{\alpha s} u_{\alpha s}^\mu. \quad (15)$$

where,

$$\mathcal{R}_\alpha(IQ) = [I_\alpha Q_\alpha + I_\alpha Q_{\alpha 0} + I_{\alpha 0} Q_\alpha + I_{\alpha 0} Q_{\alpha 0} \mathbb{I}]. \quad (16)$$

Thus $\text{tr} [n_\alpha u_\alpha^\mu] = [\text{tr} (IQ) + N I_{\alpha 0} Q_{\alpha 0}] n_{\alpha s} u_{\alpha s}^\mu$ and $\text{tr} (IQ) = N I_b Q^b$, where the summation convention over the colour index b is implied. Thus one can write expression for the colour density as

$$j_\alpha^\mu = \frac{g}{2} [\overline{I_\alpha Q_\alpha} + I_{\alpha 0} Q_\alpha + I_\alpha Q_{\alpha 0}] n_{\alpha s} u_{\alpha s}^\mu \quad (17)$$

where, $\overline{I_\alpha Q_\alpha} = I_\alpha Q_\alpha - \frac{1}{N} \text{tr} (I_\alpha Q_\alpha)$ is a trace-less quantity. It is easy to check that $\text{tr} (j_\alpha^\mu) = 0$ as it should be the case. We have replaced of n_α and u_α^μ with a new (and more) set of variables $(I_\alpha + I_{\alpha 0}), (Q_\alpha + Q_{\alpha 0}) n_{\alpha s}$ and

$u_{\alpha s}^\mu$. Since we have introduced two more variables than the originals, we have a freedom to impose two conditions on them. Before exercising this freedom, let us first note that colour singlet flux $n_{\alpha s} u_{\alpha s}^\mu$ may be a conserved quantity in absence of any collisions i.e.,

$$\partial_\mu (n_{\alpha s} u_{\alpha s}^\mu) = 0. \quad (18)$$

After substituting for j_α^μ into $D_\mu j_\alpha^\mu = 0$ and after using

$$\{F_\mu^\nu, n_\alpha u_\alpha^\mu\} = \left\{ T^d, j_\alpha^\mu + \left(\frac{1}{N} \text{tr} (I_\alpha Q_\alpha) \mathbb{I} + I_{\alpha 0} Q_{\alpha 0} \mathbb{I} \right) n_{\alpha s} u_{\alpha s}^\mu \right\} F_{d\mu}^\nu \quad (20)$$

where, we have used $F_\mu^\nu = F_{d\mu}^\nu T^d$. From this one can find

$$\text{tr} \{F_\mu^\nu, n_\alpha u_\alpha^\mu\} = 2 j_{\alpha b}^\mu F_{b\mu}^\nu \quad (21)$$

$$t_\alpha^{\mu\nu} = 2 \left[\overline{I_\alpha Q_\alpha} + I_\alpha Q_{\alpha 0} + I_{\alpha 0} Q_\alpha + \frac{1}{N} \text{tr} (I_\alpha Q_\alpha) + I_{\alpha 0} Q_{\alpha 0} \right] (Q_\alpha + Q_{\alpha 0}) n_{\alpha s} u_{\alpha s}^\mu u_{\alpha s}^\nu \quad (22)$$

Next we can exercise our freedom and impose the following conditions:

$$u_{\alpha s}^\mu D_\mu (I_\alpha) = 0 \quad (23)$$

$$u_{\alpha s}^\mu D_\mu (Q_\alpha) = 0 \quad (24)$$

From this it is easy to see that $I_{\alpha\alpha}^2$ and $Q_{\alpha\alpha}^2$ are the constants of motions and one may identify $I_{\alpha 0} = \sqrt{I_{\alpha\alpha}^2}$ and $Q_{\alpha 0} = \sqrt{Q_{\alpha\alpha}^2}$. It is worth noting that dynamical

$$D_\mu t_\alpha^{\mu\nu} = 2 \left[\overline{I_\alpha Q_\alpha} + I_\alpha Q_{\alpha 0} + I_{\alpha 0} Q_\alpha + \frac{1}{N} \text{tr} (I_\alpha Q_\alpha) \mathbb{I} + I_{\alpha 0} Q_{\alpha 0} \mathbb{I} \right] (Q_\alpha + Q_{\alpha 0}) \partial_\mu (n_{\alpha s} u_{\alpha s}^\mu u_{\alpha s}^\nu) \quad (26)$$

By taking trace of $t_\alpha^{\mu\nu}$ expression and equating it with

equations (16-17) one gets,

$$u_{\alpha s}^\mu D_\mu [\overline{I Q} + I_\alpha Q_{\alpha 0} + I_{\alpha 0} Q_\alpha] = 0 \quad (19)$$

This equation has similar structure with the equation of colour charge dynamics given by equation (3).

Next consider the "Lorentz-force" term described by the right hand-side of equation (7), by adding and subtracting terms $\frac{1}{N} \text{tr} (I Q)$ in $n_\alpha u_{\alpha s}^\mu$ one can write

where, we have used $T^d j_\alpha^\mu = \frac{1}{N} \delta^{db} j_{\alpha b}^\mu \mathbb{I}$. Thus the trace of the "Lorentz-force" term may help us in identifying $[\overline{I_\alpha Q_\alpha} + I_{\alpha 0} Q_\alpha + I_\alpha Q_{\alpha 0}]$ as the colour charge similar [6] and it obeys the same differential equation [equation (3)].

Next, consider $t_\alpha^{\mu\nu}$ in terms of the new variables

equations for both I and Q are having the same form as the colour-charge dynamics equation of Ref.[6]. After the imposition of the above conditions, equation (18) can be written as

$$u_{\alpha s}^\mu D_\mu (\overline{I Q}) = 0. \quad (25)$$

The covariant derivative of $t_\alpha^{\mu\nu}$ can be written as

(20) we get

$$\mathcal{N} n_{\alpha s} u_{\alpha s}^\mu \partial_\mu u_\alpha^\mu = g j_{\alpha b}^\mu F_{b\mu}^\nu \quad (27)$$

where, \mathcal{N} is a function of time in general and it is defined a trace of the following expression:

$$\mathcal{N} = \text{tr} \left\{ \left[\overline{I_\alpha Q_\alpha} + I_\alpha Q_{\alpha 0} + I_{\alpha 0} Q_\alpha + \frac{1}{N} \text{tr} (I_\alpha Q_\alpha) \mathbb{I} + I_{\alpha 0} Q_{\alpha 0} \mathbb{I} \right] (Q_\alpha + Q_{\alpha 0}) \right\}. \quad (28)$$

It should be noted that \mathcal{N} is a function of space and time

in general. But one can note that N can be calculated

using dynamics of I_α and Q_α given by equations (22-23). Therefore if at some initial time t_0 if $I_\alpha(t_0) = Q_\alpha(t_0)$

However it can be simplified if one notes that the dynamics of Q_α and I_α are being described by the similar kind of equations [i.e. equations (22-23)]. Therefore if at some initial time their values are taken to be same then for all the subsequent times they will remain same i.e. $I_\alpha(t) = Q_\alpha(t)$. Under these conditions $\mathcal{N} = 4(Q_{\alpha a}^2)^{3/2}$ can become a constant. This would allow for the redefinition of the colour charge in equations (18-21) by dividing it by the factor $[I_\alpha Q_\alpha + I_{\alpha 0} Q_\alpha + I_\alpha Q_{\alpha 0}]$ appearing in equations (18, 26) by $4(Q_{\alpha a}^2)^{3/2}$. Thus we have shown that when the charge associated with the charge density ρ_α i.e. I_α and that with the matrix velocity U_α^μ i.e. Q_α are set to be same at some initial time, then the both hydrodynamics we discussed here become equivalent.

In conclusion, we have discussed the relationship between the two versions of the equations of colour hydrodynamics. We have shown that under the assumption that the charges associated with density and velocity variables of the matrix hydrodynamics are same at some initial time then the both versions become identical. However, in deriving we have ignored pressure gradients for the simplicity. But terms with the pressure gradients can be incorporated by some straightforward arguments.
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* jeet@prl.res.in

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